ISRAEL JOURNAL OF MATHEMATICS 90 (1995), 67–79

ON BIDETERMINANTAL FORMULAS FOR CHARACTERS OF CLASSICAL GROUPS

BY

C. KRATTENTHALER

Institut für Mathematik der Universität Wien
Strudlhofgasse 4, A-1090 Wien, Austria
e-mail: KRATT@Pap.Univie.Ac.At

ABSTRACT

New bideterminantal formulas for the irreducible symplectic and orthogonal characters are given that generalize the classical bideterminantal formulas. These formulas are analogous to Regev's (Israel J. Math. 80 (1992), 155–160) bideterminantal formulas for Schur functions, the irreducible general linear characters. Also, new bideterminantal formulas for Proctor's intermediate symplectic characters are derived.

1. Introduction and statement of results

We prove formulas that express the irreducible symplectic and orthogonal characters (cf. [1, 2, 3]) as ratios of certain determinants whose entries involve complete homogeneous, respectively elementary symmetric functions (see (1.8)–(1.13) below). These new identities generalize the classical bideterminantal expressions for irreducible symplectic and orthogonal characters (cf. [1, § 24.2]). We were motivated by work of Regev [8] who proved analogous formulas for Schur functions, the irreducible general linear characters.

Let us recall the classical character formulas. An n-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ is called a **partition**. The components of λ are called the **parts** of λ. The Schur function $s_\lambda(x_1, x_2, \ldots, x_n)$ is given by (cf. [4, I.(3.1); 1, p. 403, (A.4)])

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det\limits_{1\leq i,j\leq n}\left(x_j^{\lambda_i+n-i}\right)}{\det\limits_{1\leq i,j\leq n}\left(x_j^{n-i}\right)}. \tag{1.1}$$

Received September 7, 1993 and in revised form December 14, 1993

The irreducible (even) symplectic character $\mathrm{sp}_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1})$ is given by (cf. [1, (24.18)])

$$\mathrm{sp}_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1}) = \frac{\det\limits_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i+1} - x_j^{-(\lambda_i+n-i+1)}\right)}{\det\limits_{1\le i,j\le n}\left(x_j^{n-i+1} - x_j^{-(n-i+1)}\right)}. \tag{1.2}$$

Odd symplectic characters have been defined by Proctor [5, 6, 7]. These correspond to indecomposable representations of odd symplectic groups. A bideterminantal formula for the odd symplectic character $\mathrm{sp}_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1}, 1)$ appeared in [5, Theorem 2.2]. These characters are indexed by partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})$ with $n+1$ parts. Proctor's formula may be written as

$$\mathrm{sp}_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1}, 1) = \left.\frac{\det\limits_{1\le i,j\le n+1}\left(x_j^{\lambda_i+n-i+3/2} + x_j^{-(\lambda_i+n-i+3/2)}\right)}{\det\limits_{1\le i,j\le n+1}\left(x_j^{n-i+3/2} + x_j^{-(n-i+3/2)}\right)}\right|_{x_{n+1}=1}. \tag{1.3}$$

Now let $\lambda = (\lambda_1, \dots, \lambda_n)$ again be a partition with n parts. For the even orthogonal groups, the irreducible orthogonal character $o_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1})$ is given by (cf. [1, (24.40) + first paragraph on p. 411, 7, Appendix A2])

$$o_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1}) = \frac{\det\limits_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i} + x_j^{-(\lambda_i+n-i)}\right)}{\det\limits_{1\le i,j\le n}\left(x_j^{n-i} + x_j^{-(n-i)}\right)}. \tag{1.4}$$

Finally, for the odd orthogonal groups, the irreducible orthogonal character $o_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1}, 1)$ is given by (cf. [1, (24.28)])

$$o_\lambda(x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1}, 1) = \frac{\det\limits_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i+1/2} - x_j^{-(\lambda_i+n-i+1/2)}\right)}{\det\limits_{1\le i,j\le n}\left(x_j^{n-i+1/2} - x_j^{-(n-i+1/2)}\right)}. \tag{1.5}$$

Let $h_m(x_1, \dots, x_r)$ denote the m-th complete homogeneous symmetric function in the variables $x_1, x_2, \dots, x_r$, and let $e_m(x_1, \dots, x_r)$ denote the m-th elementary symmetric function in $x_1, x_2, \dots, x_r$, (cf. [4, pp. 12–15]). Then Regev's formulas for Schur functions s_λ read as follows, [8, Theorem 1.(b)]:

$$s_\lambda(x_1, \dots, x_{n+r}) = \frac{\det\limits_{1\le i,j\le n}\left(h_{\lambda_i+n-i}(x_j, \dots, x_{j+r})\right)}{\det\limits_{1\le i,j\le n}\left(h_{n-i}(x_j, \dots, x_{j+r})\right)}, \tag{1.6}$$

and if $n-2 \le r$ [8, Theorem 1'.(b), corrected]

$$(1.7) \qquad s_{\lambda'}(x_n,\dots,x_{r+1}) = \frac{\det\limits_{1\le i,j\le n}\left(e_{\lambda_i+n-i}(x_j,\dots,x_{j+r})\right)}{\det\limits_{1\le i,j\le n}\left(e_{n-i}(x_j,\dots,x_{j+r})\right)}.$$

Here λ' denotes the partition conjugate to λ (cf. [4, p. 2]). Clearly, setting $r=0$ in (1.6) yields (1.1).

The analogous identities for sp_λ and o_λ which we are going to prove in the next section, are the following. For the symplectic characters we have

$$(1.8) \qquad \mathrm{sp}_\lambda(x_1^{\pm 1},\dots,x_{n+r}^{\pm 1}(,1)) = \frac{\det\limits_{1\le i,j\le n}\left(h_{\lambda_i+n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))\right)}{\det\limits_{1\le i,j\le n}\left(h_{n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))\right)},$$

and if $n-2\le r$,

$$(1.9) \qquad \mathrm{sp}_{\lambda'}(x_n^{\pm 1},\dots,x_{r+1}^{\pm 1}(,1)) = \frac{\det\limits_{1\le i,j\le n}\left(e'_{\lambda_i+n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))\right)}{\det\limits_{1\le i,j\le n}\left(e'_{n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))\right)},$$

where

$$e'_m() := e_m() - e_{m-2}().$$

By $h_m(x_1^{\pm 1},\dots,x_r^{\pm 1}(,1))$ we mean $h_m(x_1,x_1^{-1},\dots,x_r,x_r^{-1}(,1))$, etc. The notation $(,1)$ has to be understood in the sense that this group has to be omitted in case of even symplectic characters, while in case of odd symplectic characters it is meant that 1 is an additional argument in the symplectic character and the complete homogenous and the elementary symmetric functions in (1.8) and (1.9). Obviously, the identity (1.2) for the even symplectic character comes out of the even case of (1.8) by setting $r=0$ and multiplying the j-th column in both determinants by $x_j - x_j^{-1}$, for $j=1,2,\dots,n$. The identity (1.3) cannot be derived in full generality from (1.8). The reason is that in (1.3) the partition λ is allowed to have $n+1$ parts which is not true for (1.8). However, if $\lambda_{n+1}=0$ then (1.3) and the odd case of (1.8) with $r=0$ are completely equivalent. This will be shown at the end of section 2.

Similarly, for the orthogonal characters we shall prove

$$(1.10) \qquad o_\lambda(x_1^{\pm 1},\dots,x_{n+r}^{\pm 1}(,1)) = \frac{\det\limits_{1\le i,j\le n}\left(h'_{\lambda_i+n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))\right)}{\det\limits_{1\le i,j\le n}\left(h'_{n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))\right)},$$

where $h'_m() := h_m() - h_{m-2}()$, and, if $n-2 \le r$,

$$o_{\lambda'}(x_n^{\pm 1}, \dots, x_{r+1}^{\pm 1}(,1)) = \frac{\det\limits_{1\le i,j\le n}\left(e_{\lambda_i+n-i}(x_j^{\pm 1}, \dots, x_{j+r}^{\pm 1}(,1))\right)}{\det\limits_{1\le i,j\le n}\left(e_{n-i}(x_j^{\pm 1}, \dots, x_{j+r}^{\pm 1}(,1))\right)}. \tag{1.11}$$

Also here the terms $(,1)$ in (1.10) and (1.11) are optional, depending on whether o_λ is meant to be the even or odd orthogonal character. Again, (1.4) (the even case) comes out of (1.10) by setting $r=0$ while (1.5) (the odd case) comes out of (1.10) by setting $r=0$ and multiplying the j-th column in both determinants by $x_j^{1/2} - x_j^{-1/2}$, for $j = 1, 2, \dots, n$.

Identities (1.6)–(1.11) allow a unified formulation, which we give in the following theorem.

THEOREM 1: *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition and let $\chi_\lambda(x_1, x_2, \dots, x_m)$ be any one of the characters $s_\lambda(x_1, x_2, \dots, x_m)$, $\mathrm{sp}_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}(,1))$, or $o_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}(,1))$. Then there hold*

$$\chi_\lambda(x_1, x_2, \dots, x_{n+r}) = \frac{\det\limits_{1\le i,j\le n}\left(\chi_{(\lambda_i+n-i)}(x_j, x_{j+1}, \dots, x_{j+r})\right)}{\det\limits_{1\le i,j\le n}\left(\chi_{(n-i)}(x_j, x_{j+1}, \dots, x_{j+r})\right)}, \tag{1.12}$$

and, if $n-2 \le r$,

$$\chi_{\lambda'}(x_n, x_{n+1}, \dots, x_{r+1}) = \frac{\det\limits_{1\le i,j\le n}\left(\chi_{(1^{\lambda_i+n-i})}(x_j, x_{j+1}, \dots, x_{j+r})\right)}{\det\limits_{1\le i,j\le n}\left(\chi_{(1^{n-i})}(x_j, x_{j+1}, \dots, x_{j+r})\right)}. \tag{1.13}$$

Regev notes that the determinants in the denominators of (1.6) and (1.7) factor. Namely, [8, Theorem 1.(a)]

$$\det_{1\le i,j\le n}\left(h_{n-i}(x_j, \dots, x_{j+r})\right) = \prod_{1\le i<j\le n} (x_i - x_{j+r}) \tag{1.14}$$

and [8, Theorem 1'.(a)]

$$\det_{1\le i,j\le n}\left(e_{n-i}(x_j, \dots, x_{j+r})\right) = \prod_{1\le j\le i\le n-1} (x_i - x_{i-j+r+2}). \tag{1.15}$$

Also the determinants in the denominators of (1.8)–(1.11) factor. Namely we have

$$\begin{aligned}\det_{1\le i,j\le n}\left(h_{n-i}(x_j^{\pm 1}, \dots, x_{j+r}^{\pm 1})\right) &= \det_{1\le i,j\le n}\left(h'_{n-i}(x_j^{\pm 1}, \dots, x_{j+r}^{\pm 1}(,1))\right)\\ &= \prod_{1\le i<j\le n} (x_i - x_{j+r})\Big(1 - \frac{1}{x_i x_{j+r}}\Big)\end{aligned} \tag{1.16}$$

and

$$\det_{1\le i,j\le n}\left(e'_{n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1})\right)=\det_{1\le i,j\le n}\left(e_{n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))\right)$$
$$(1.17)\qquad =\prod_{1\le j\le i\le n-1}(x_i-x_{i-j+r+2})(1-\frac{1}{x_i x_{i-j+r+2}}).$$

In the next section we first give the corrected proof of (1.7) (which is completely analogous to the proof of (1.6) in [8]). Subsequently, by following the same idea, we prove (1.8)–(1.11), and thus (1.12) and (1.13), in a uniform fashion. The proofs of (1.16) and (1.17) are also contained in the next section. In section 3 we give a generalization of Theorem 1, which for the special case of Schur functions already appeared in Regev's paper [8, p. 159]. From this generalization (Theorem 2) we derive more bideterminantal formulas for odd orthogonal characters and bideterminantal formulas for Proctor's [5, 6] intermediate symplectic characters that interpolate between symplectic characters and Schur functions.

2. The proofs

For a proof of (1.7) we start by defining three $n\times n$ matrices, H_λ, B_λ, and $M^{(r)}$,

$$\begin{aligned} H_\lambda &:= (e_{\lambda_i-i+j}(x_n,\dots,x_{r+1}))_{1\le i,j\le n},\\ B_\lambda &:= (e_{\lambda_i+n-i}(x_j,\dots,x_{j+r}))_{1\le i,j\le n},\\ M^{(r)} &:= (e_{n-i}(x_j,\dots,x_{n-1},x_{r+2},\dots,x_{j+r}))_{1\le i,j\le n}.\end{aligned}$$

We claim that there holds

$$(2.1)\qquad B_\lambda = H_\lambda\cdot M^{(r)}.$$

In order to show this identity, we have to verify

$$(2.2)\quad e_{\lambda_i+n-i}(x_j,\dots,x_{j+r}) = \sum_{k=1}^{n} e_{\lambda_i-i+k}(x_n,\dots,x_{r+1})\,e_{n-k}(x_j,\dots,x_{n-1},x_{r+2},\dots,x_{j+r}).$$

Now, the generating function for elementary symmetric functions is

$$(2.3)\qquad \sum_{m=0}^{\infty} e_m(y_1,\dots,y_s)z^m=\prod_{i=1}^{s}(1+y_iz).$$

Therefore, by comparing the coefficients of z^m on both sides of

$$\prod_{i=j}^{j+r}(1+x_iz)=\prod_{i=n}^{r+1}(1+x_iz)\cdot\left(\prod_{i=j}^{n-1}(1+x_iz)\prod_{i=r+2}^{j+r}(1+x_iz)\right)$$

we obtain

(2.4)
$$e_m(x_j,\dots,x_{j+r})=\sum_{p=0}^{m}e_{m-p}(x_n,\dots,x_{r+1})\,e_p(x_j,\dots,x_{n-1},x_{r+2},\dots,x_{j+r}).$$

Since the number of arguments in $e_p(x_j,\dots,x_{n-1},x_{r+2},\dots,x_{j+r})$ is $n-1$, the summands at the right-hand side of (2.4) vanish for $p\geq n$. Hence, we may replace p by $n-k$, where k runs from 1 to n, thus obtaining (2.2) when we also set $m=\lambda_i+n-i$. This establishes (2.1).

Next, on both sides of (2.1) we take determinants. By the dual Jacobi–Trudi identity for Schur functions (cf. [4, I.(3.5); 1, (24.11), (A.6)]) we have $\det H_\lambda = s_{\lambda'}(x_n,\dots,x_{r+1})$. By setting $\lambda=0$ in (2.1) we deduce $\det M^{(r)}=\det B_0=\det(e_{n-i}(x_j,\dots,x_{j+r}))$. Hence, (1.7) is established. ∎

The proofs of (1.8)–(1.11) are analogous, only the choice of the matrices $H_\lambda, B_\lambda, M^{(r)}$ and the Jacobi–Trudi type identity that is used differ. So it will suffice to give a detailed proof only for the even case of (1.8). For the odd case of (1.8) and for (1.9)–(1.11) we only state the choices of the matrices $H_\lambda, B_\lambda, M^{(r)}$ and give the respective references for the Jacobi–Trudi type identities.

In order to prove the even case of (1.8) we choose

$$\begin{aligned}H_\lambda &= \big(h_{\lambda_i-i+1}(x_1^{\pm1},\dots,x_{n+r}^{\pm1}) \;\vdots\; h_{\lambda_i-i+j}(x_1^{\pm1},\dots,x_{n+r}^{\pm1})\\ &\qquad\qquad + h_{\lambda_i-i-j+2}(x_1^{\pm1},\dots,x_{n+r}^{\pm1})\big)_{1\leq i,j\leq n},\\ B_\lambda &= \big(h_{\lambda_i+n-i}(x_j^{\pm1},\dots,x_{j+r}^{\pm1})\big)_{1\leq i,j\leq n},\\ M^{(r)} &= \big((-1)^{n-i}e_{n-i}(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1})\big)_{1\leq i,j\leq n}.\end{aligned}$$

The notation for H_λ means that the first expression gives the entries of the first column while the second expression gives the entries of the remaining columns, $j\geq 2$. Again we claim that

$$B_\lambda = H_\lambda\cdot M^{(r)}. \tag{2.5}$$

Here we have to verify

$$\begin{aligned}h_{\lambda_i+n-i}(x_j^{\pm1},\dots,x_{j+r}^{\pm1}) &= h_{\lambda_i-i+1}(x_1^{\pm1},\dots,x_{n+r}^{\pm1})\cdot(-1)^{n-1}\\ &\qquad\cdot e_{n-1}(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1})\\ &\quad+\sum_{k=2}^{n}(h_{\lambda_i-i+k}(x_1^{\pm1},\dots,x_{n+r}^{\pm1})+h_{\lambda_i-i-k+2}(x_1^{\pm1},\dots,x_{n+r}^{\pm1}))\\ &\qquad\cdot(-1)^{n-k}e_{n-k}(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1}).\end{aligned}\tag{2.6}$$

The generating function for the complete homogeneous symmetric functions is

$$\sum_{m=0}^{\infty}h_m(y_1,\dots,y_s)z^m=\prod_{i=1}^{s}\frac{1}{(1-y_iz)}.\tag{2.7}$$

Therefore, by comparing the coefficients of z^m on both sides of

$$\begin{aligned}\prod_{i=j}^{j+r}\frac{1}{(1-x_iz)(1-x_i^{-1}z)}&=\prod_{i=1}^{n+r}\frac{1}{(1-x_iz)(1-x_i^{-1}z)}\\ &\qquad\left(\prod_{i=1}^{j-1}(1-x_iz)(1-x_i^{-1}z)\prod_{i=j+r+1}^{n+r}(1-x_iz)(1-x_i^{-1}z)\right)\end{aligned}$$

we get

$$\begin{aligned}h_m(x_j^{\pm1},\dots,x_{j+r}^{\pm1})&=\sum_{p=0}^{m}h_{m-p}(x_1^{\pm1},\dots,x_{n+r}^{\pm1})(-1)^p\\ &\qquad\cdot e_p(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1}).\end{aligned}\tag{2.8}$$

The number of arguments in $e_p(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1})$ is $2n-2$. Hence the summands at the right-hand side of (2.8) vanish for $p\ge 2n-1$. Besides, we have

$$e_p(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1})=e_{2n-2-p}(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1}),$$

since the product of all $2n-2$ arguments equals 1 and the set of arguments is invariant under taking inverses. So, if we reorder the summands at the right-hand side of (2.8) by singling out the $(n-1)$-st summand and pairing the p-th with the $(2n-2-p)$-th summand, for $p=n-2,n-3,\dots,1,0$, after replacing m by λ_i+n-i we obtain (2.6). This establishes (2.5). The remaining steps are the same as before in the proof of (1.7). This time $\det H_\lambda$ is identified to be the symplectic character $\mathrm{sp}_\lambda(x_1^{\pm 1},\dots,x_{n+r}^{\pm 1})$ by virtue of the symplectic Jacobi–Trudi identity (cf. [1, Prop. 24.22]). ∎

The odd case of (1.8) is established in the same manner. We choose

$$\begin{aligned} H_\lambda &= \big(h_{\lambda_i-i+1}(x_1^{\pm1},\dots,x_{n+r}^{\pm1},1) \ \vdots \ h_{\lambda_i-i+j}(x_1^{\pm1},\dots,x_{n+r}^{\pm1},1) \\ &\qquad + h_{\lambda_i-i-j+2}(x_1^{\pm1},\dots,x_{n+r}^{\pm1},1)\big)_{1\le i,j\le n}, \\ B_\lambda &= \big(h_{\lambda_i+n-i}(x_j^{\pm1},\dots,x_{j+r}^{\pm1},1)\big)_{1\le i,j\le n}, \\ M^{(r)} &= \big((-1)^{n-i}e_{n-i}(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1})\big)_{1\le i,j\le n}. \end{aligned}$$

The Jacobi–Trudi type identity for the odd symplectic character that has to be used in this instance is [5, p. 317]

$$\begin{aligned} &\mathrm{sp}_\lambda(x_1^{\pm1},x_2^{\pm1},\dots,x_n^{\pm1},1) \\ &\quad = \det_{1\le i,j\le n}\big(h_{\lambda_i-i+1}(x_1^{\pm1},\dots,x_n^{\pm1},1) \ \vdots \ h_{\lambda_i-i+j}(x_1^{\pm1},\dots,x_n^{\pm1},1) \\ &\qquad + h_{\lambda_i-i-j+2}(x_1^{\pm1},\dots,x_n^{\pm1},1)\big). \end{aligned}$$ ∎

In order to prove (1.9) we choose

$$\begin{aligned} H_\lambda &= \big(e'_{\lambda_i-i+1}(x_n^{\pm1},\dots,x_{r+1}^{\pm1}(,1)) \ \vdots \ e'_{\lambda_i-i+j}(x_n^{\pm1},\dots,x_{r+1}^{\pm1}(,1)) \\ &\qquad + e'_{\lambda_i-i-j+2}(x_n^{\pm1},\dots,x_{r+1}^{\pm1}(,1))\big)_{1\le i,j\le n}, \\ B_\lambda &= \big(e'_{\lambda_i+n-i}(x_j^{\pm1},\dots,x_{j+r}^{\pm1}(,1))\big)_{1\le i,j\le n}, \\ M^{(r)} &= \big(e_{n-i}(x_j^{\pm1},\dots,x_{n-1}^{\pm1},x_{r+2}^{\pm1},\dots,x_{j+r}^{\pm1})\big)_{1\le i,j\le n}. \end{aligned}$$

That $\det H_\lambda$ equals $\mathrm{sp}_{\lambda'}(x_n^{\pm1},\dots,x_{r+1}^{\pm1}(,1))$ is due to the dual symplectic Jacobi–Trudi identities (cf. [1, Cor. 24.24] for the even case and [7, Appendix A2] for the odd case). ∎

For the proof of (1.10) we choose

$$\begin{aligned} H_\lambda &= \big(h'_{\lambda_i-i+1}(x_1^{\pm1},\dots,x_{n+r}^{\pm1}(,1)) \ \vdots \ h'_{\lambda_i-i+j}(x_1^{\pm1},\dots,x_{n+r}^{\pm1}(,1)) \\ &\qquad + h'_{\lambda_i-i-j+2}(x_1^{\pm1},\dots,x_{n+r}^{\pm1}(,1))\big)_{1\le i,j\le n}, \\ B_\lambda &= \big(h'_{\lambda_i+n-i}(x_j^{\pm1},\dots,x_{j+r}^{\pm1}(,1))\big)_{1\le i,j\le n}, \\ M^{(r)} &= \big((-1)^{n-i}e_{n-i}(x_1^{\pm1},\dots,x_{j-1}^{\pm1},x_{j+r+1}^{\pm1},\dots,x_{n+r}^{\pm1})\big)_{1\le i,j\le n}. \end{aligned}$$

The orthogonal Jacobi–Trudi identities (cf. [1, Prop. 24.44, Prop. 24.33]) guarantee that $\det H_\lambda$ equals $o_\lambda(x_1^{\pm 1},\dots,x_{n+r}^{\pm 1}(,1))$. ∎

Finally, for proving (1.11) we choose

$$\begin{aligned} H_\lambda &= (e_{\lambda_i-i+1}(x_n^{\pm 1},\dots,x_{r+1}^{\pm 1}(,1)) \ \vdots\ e_{\lambda_i-i+j}(x_n^{\pm 1},\dots,x_{r+1}^{\pm 1}(,1)) \\ &\qquad\qquad + e_{\lambda_i-i-j+2}(x_n^{\pm 1},\dots,x_{r+1}^{\pm 1}(,1)))_{1\le i,j\le r}, \\ B_\lambda &= (e_{\lambda_i+n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1)))_{1\le i,j\le r}, \\ M^{(r)} &= (e_{n-i}(x_j^{\pm 1},\dots,x_{n-1}^{\pm 1},x_{r+2}^{\pm 1},\dots,x_{j+r}^{\pm 1}))_{1\le i,j\le r}. \end{aligned}$$

Here we use the dual orthogonal Jacobi–Trudi identities (cf. [1, Cor. 24.45, Cor. 24.35]). ∎

Now we turn to the determinant factorizations (1.16) and (1.17). What we do is to change Regev's proofs of (1.14) and (1.15) into proofs of (1.16) and (1.17). First we have to find analogues for Lemmas 5 and 5' (bottom line of p.158) in Regev's paper. These analogues can be given in a unified form. They read

$$\begin{aligned} (2.9)\quad & \tilde h_s(x_1^{\pm 1},\dots,x_v^{\pm 1}(,1)) - \tilde h_s(x_2^{\pm 1},\dots,x_{v+1}^{\pm 1}(,1)) \\ &\qquad = (x_1-x_{v+1})(1-\frac{1}{x_1x_{v+1}})\tilde h_{s-1}(x_1^{\pm 1},\dots,x_{v+1}^{\pm 1}(,1)), \end{aligned}$$

where $\tilde h_m$ is any of h_m or h'_m, and

$$\begin{aligned} (2.10)\quad & \tilde e_s(x_1^{\pm 1},\dots,x_v^{\pm 1}(,1)) - \tilde e_s(x_2^{\pm 1},\dots,x_{v+1}^{\pm 1}(,1)) \\ &\qquad = (x_1-x_{v+1})(1-\frac{1}{x_1x_{v+1}})\tilde e_{s-1}(x_2^{\pm 1},\dots,x_v^{\pm 1}(,1)), \end{aligned}$$

where $\tilde e_m$ is any of e_m or e'_m. These two identities are easily proved by means of generating functions. For a proof of (1.16), let $\det(n,r)$ denote the determinant

$$\det_{1\le i,j\le n}(\tilde h_{n-i}(x_j^{\pm 1},\dots,x_{j+r}^{\pm 1}(,1))).$$

The last row of this determinant consists of all 1's. Subtract the j-th column from the $(j-1)$-st column, $j=2,\dots,n$, and then expand in the last row. By using the relation (2.9) we obtain

$$\det(n,r) = \det(n-1,r+1)\prod_{j=1}^{n-1}(x_j-x_{1+j+r})\Big(1-\frac{1}{x_jx_{j+1+r}}\Big).$$

By induction on n we are done. The proof of (1.17) is similar. ∎

At the end of this section we show that (1.3) with $\lambda_{n+1}=0$ and the odd case of (1.8) with $r=0$ are equivalent. In order to see this, first observe that because of $x_{n+1}=1$ in both determinants in (1.3) the last column consists of all 2's. Hence, if we subtract the last row from all the other rows in both determinants in (1.3), the last column will consist of only zeros with the exception of a 2 in the last row. Expansion in the last column gives

$$\operatorname{sp}_\lambda(x_1^{\pm 1},x_2^{\pm 1},\dots,x_n^{\pm 1},1)$$
$$=\frac{\det\limits_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i+3/2}+x_j^{-(\lambda_i+n-i+3/2)}-x_j^{1/2}-x_j^{-1/2}\right)}{\det\limits_{1\le i,j\le n}\left(x_j^{n-i+3/2}+x_j^{-(n-i+3/2)}-x_j^{1/2}-x_j^{-1/2}\right)}.$$

Now in both determinants we divide the j-th column by $x_j^{3/2}+x_j^{-3/2}-x_j^{1/2}-x_j^{-1/2}$ and obtain (1.8) with $r=0$ by using the identity

$$h_m(x,x^{-1},1)=\frac{x^{m+3/2}+x^{-(m+3/2)}-x^{1/2}-x^{-1/2}}{x^{3/2}+x^{-3/2}-x^{1/2}-x^{-1/2}}.$$

That this identity actually holds is easily verified by using generating functions. ∎

3. A generalization

Here we are going to state a generalization of Theorem 1. Let $\lambda=(\lambda_1,\dots,\lambda_n)$ be a partition and let $\mathbf{z}$ denote the sequence $z_1,z_2,\dots,z_m$ of variables. We formally define

$$\operatorname{sp}_\lambda(x_1^{\pm 1},x_2^{\pm 1},\dots,x_n^{\pm 1},z_1,\dots,z_m)=\operatorname{sp}_\lambda(x_1^{\pm 1},x_2^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z})$$

(3.1a)
$$=\det_{1\le i,j\le n}\Big(h_{\lambda_i-i+1}(x_1^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z})\ \vdots\ h_{\lambda_i-i+j}(x_1^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z}) + h_{\lambda_i-i-j+2}(x_1^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z})\Big)$$

(3.1b)
$$=\det_{1\le i,j\le n}\Big(e'_{\lambda'_i-i+1}(x_1^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z})\ \vdots\ e'_{\lambda'_i-i+j}(x_1^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z}) + e'_{\lambda'_i-i-j+2}(x_1^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z})\Big)$$

and

$$o_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, z_1, \dots, z_m) = o_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z})$$

(3.2a)
$$= \det_{1\le i,j\le n}\left(h'_{\lambda_i-i+1}(x_1^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z}) \;\vdots\; h'_{\lambda_i-i+j}(x_1^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z}) + h'_{\lambda_i-i-j+2}(x_1^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z})\right)$$

(3.2b)
$$= \det_{1\le i,j\le n}\left(e_{\lambda'_i-i+1}(x_1^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z}) \;\vdots\; e_{\lambda'_i-i+j}(x_1^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z}) + e_{\lambda'_i-i-j+2}(x_1^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z})\right).$$

The notation $h_k(x_1^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z})$ means $h_k(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, z_1, \dots, z_m)$, and similarly for e_k, h'_k, and e'_k. That the determinants in (3.1) agree and that the determinants in (3.2) agree is well-known (see [1, Prop. A.44, Cor. A.46] for a proof). The expressions $\mathrm{sp}_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}, \mathbf{z})$ and $o_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}, \mathbf{z})$ are sometimes called **formal** characters.

Now we are able to formulate the announced generalization of Theorem 1.

Theorem 2: *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition and let $\chi_\lambda(x_1, x_2, \dots, x_m)$ be any one of the characters $s_\lambda(x_1, x_2, \dots, x_m, \mathbf{z}), \mathrm{sp}_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}, \mathbf{z})$, or $o_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_m^{\pm 1}, \mathbf{z})$. Then there hold*

(3.3)
$$\chi_\lambda(x_1, x_2, \dots, x_{n+r}) = \frac{\det_{1\le i,j\le n}\left(\chi_{(\lambda_i+n-i)}(x_j, x_{j+1}, \dots, x_{j+r})\right)}{\det_{1\le i,j\le n}\left(\chi_{(n-i)}(x_j, x_{j+1}, \dots, x_{j+r})\right)},$$

and, if $n-2 \le r$,

(3.4)
$$\chi_{\lambda'}(x_n, x_{n+1}, \dots, x_{r+1}) = \frac{\det_{1\le i,j\le n}\left(\chi_{(1^{\lambda_i+n-i})}(x_j, x_{j+1}, \dots, x_{j+r})\right)}{\det_{1\le i,j\le n}\left(\chi_{(1^{n-i})}(x_j, x_{j+1}, \dots, x_{j+r})\right)}.$$

The proof of Theorem 2 is identical with the proof of Theorem 1 and is therefore omitted. Clearly, Theorem 1 is just a special case of Theorem 2. ∎

The $\mathrm{sp}_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, \mathbf{z})$ which were formally defined in (3.1) are **actual** characters. They are characters of "trace-free" representations of intermediate symplectic groups [5] that interpolate between the symplectic groups and the general linear groups. Actually, these trace-free characters are indexed by partitions with up to $n+m$ parts. So our Theorem 2 only gives bideterminantal expressions for these characters for partitions λ with at most n parts. It can be shown in a similar manner as at the end of section 2 that the $\chi_\lambda = \mathrm{sp}_\lambda$, $r=0$, $\mathbf{z}=-1$ special case of (3.3) is equivalent to the $\lambda_{n+1}=0$ case of the following bideterminantal formula for the odd symplectic character evaluated at $x_1^{\pm 1},\dots,x_n^{\pm 1},-1$,

$$\mathrm{sp}_\lambda(x_1^{\pm 1},x_2^{\pm 1},\dots,x_n^{\pm 1},-1) = \left.\frac{\det_{1\le i,j\le n+1}\left(x_j^{\lambda_i+n-i+3/2}-x_j^{-(\lambda_i+n-i+3/2)}\right)}{\det_{1\le i,j\le n+1}\left(x_j^{n-i+3/2}-x_j^{-(n-i+3/2)}\right)}\right|_{x_{n+1}=-1} \tag{3.5}$$

This identity does not appear in [5] but can be derived completely analogously as (1.3) was derived in [5, sec. 8].

A representation-theoretic meaning for the $o_\lambda(x_1^{\pm 1},x_2^{\pm 1},\dots,x_n^{\pm 1},\mathbf{z})$, formally defined in (3.2), is not so immediate. Perhaps, if intermediate orthogonal representations would be defined in analogy with the definition of intermediate symplectic representations in [5], then the expressions in (3.2) would turn out to be characters for these representations. We confine ourselves with considering (3.2) with $\mathbf{z}=-1$. This has a representation-theoretic meaning, since $o_\lambda(x_1^{\pm 1},x_2^{\pm 1},\dots,x_n^{\pm 1},-1)$ is the usual odd orthogonal character evaluated at the "negative" part of the orthogonal group. In particular, the $\chi_\lambda = o_\lambda$, $r=0$, $\mathbf{z}=-1$ special case of (3.3) is equivalent to the formula (cf. [7, proof of A2.1(d)])

$$o_\lambda(x_1^{\pm 1},x_2^{\pm 1},\dots,x_n^{\pm 1},-1) = \frac{\det_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i+1/2}+x_j^{-(\lambda_i+n-i+1/2)}\right)}{\det_{1\le i,j\le n}\left(x_j^{n-i+1/2}+x_j^{-(n-i+1/2)}\right)}. \tag{3.6}$$

References

[1] W. Fulton and J. Harris, *Representation Theory*, Springer–Verlag, New York, 1991.

[2] R. C. King, *Modification rules and products of irreducible representations of the unitary, orthogonal, and symplectic groups*, Journal of Mathematical Physics **12** (1971), 1588–1598.

[3] K. Koike and I. Terada, *Young–diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, Journal of Algebra **107** (1987), 466–511.

[4] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, New York/London, 1979.

[5] R. A. Proctor, *Odd symplectic groups*, Inventiones Mathematicae **92** (1988), 307–332.

[6] R. A. Proctor, *A generalized Berele–Schensted algorithm and conjectured Young tableaux for intermediate symplectic groups*, Transactions of the American Mathematical Society **324** (1991), 655–692.

[7] R. A. Proctor, *Young tableaux, Gelfand patterns, and branching rules for classical Lie groups*, Journal of Algebra **164** (1994), 299–360.

[8] A. Regev, *A note on Schur functions*, Israel Journal of Mathematics **80** (1992), 155–160.